Student Name: ______________________

Maths class: ______________________

James Ruse Agricultural High School

2023 **YEAR 12 Trial HSC Examination**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Pencil may be used for diagrams
- Liquid paper or white out tape is not to be used
- Calculators approved by NESA may be used
- A NESA reference sheet is provided
- In Questions 11–15, show relevant mathematical reasoning and/ or calculations

Total marks: 100

Section I – 10 marks (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6-12)

- Attempt Questions 11-15
- Allow about 2 hours 45 minutes for this section

Section I (10 marks)

Attempt Questions 1 to 10
Allow about 15 minutes for this section

Answer on the separate multiple choice answer sheet.

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1. Which of the following is not equivalent to $P \Rightarrow Q$?

(A) P is sufficient for Q

(B) Q is necessary for P

(C) If Q is false, then P is false

(D) None of the above

2. In which quadrant is the complex number $(-3+3i)^3$ located on the Argand plane?

(A) The first quadrant

(B) The second quadrant

(C) The third quadrant

(D) The fourth quadrant

3. A line has equation $\underset{\sim}{r}(t) = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$, $\lambda \in \mathbb{R}$. Which of the following is parallel to this line?

(A) $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -15 \end{pmatrix}$

(B) $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -6 \\ -15 \end{pmatrix}$

(C) $\underset{\sim}{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

(D) $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$

4. Which of the following is an expression for $\int \frac{dx}{\sqrt{7-6x-x^2}}$?

(A) $\sin^{-1}\left(\frac{x-3}{2}\right) + C$

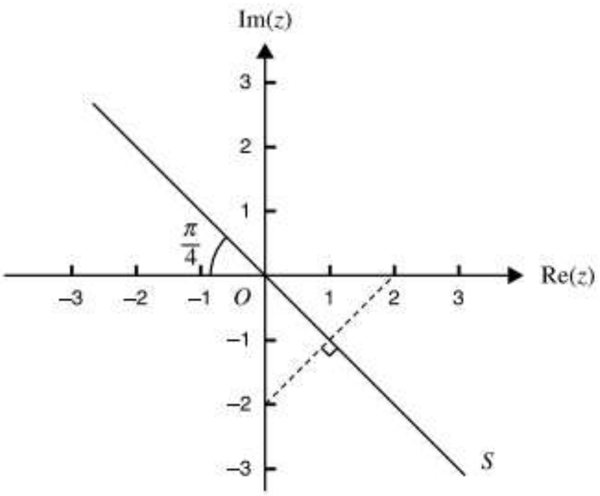
(B) $\sin^{-1}\left(\frac{x+3}{2}\right) + C$

(C) $\sin^{-1}\left(\frac{x-3}{4}\right) + C$

(D) $\sin^{-1}\left(\frac{x+3}{4}\right) + C$

5. In the diagram below, z is any complex number which lies on the line S.

Which equation best describes the locus of z?



(A) $\arg z = \frac{\pi}{4}$

(B) $\arg z = \frac{3\pi}{4}$

(C) $|z-2| = |z-2i|$

(D) $|z-2| = |z+2i|$

6. The polynomial $P(z)$ has real coefficients and $P(0) = -1$. The imaginary number α and the real number β satisfy $P(\alpha) = 0,\ P(\beta) = 0$ and $P'(\beta) = 0$.

The degree of $P(z)$ is at least:

(A) 2
(B) 3
(C) 4
(D) 5

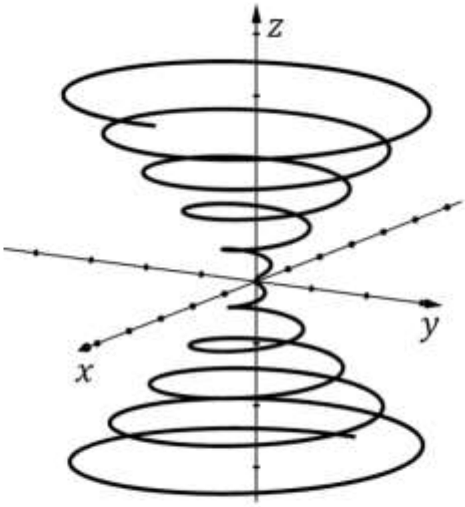
7. Which expression is equal to $\int 3\sqrt{x}\ln x\ dx$?

(A) $2x\sqrt{x}\left(\ln x - \frac{2}{3}\right) + c$
(B) $2x\sqrt{x}\left(\ln x + \frac{2}{3}\right) + c$
(C) $\frac{1}{\sqrt{x}}\left(\frac{3}{2}\ln x - 1\right) + c$
(D) $\frac{1}{\sqrt{x}}\left(\frac{3}{2}\ln x + 1\right) + c$

8. A particle is travelling in simple harmonic motion such that its velocity, in metres per second, is given by the equation $v^2 = a^2 - b^2x^2$, where $a, b \neq 0$. What is the period of motion?

(A) $\frac{2b\pi}{a}$
(B) $\frac{2a\pi}{b}$
(C) $\frac{2\pi}{a}$
(D) $\frac{2\pi}{b}$

9. Which of the equations best represent the curve below?



(A) $\underset{\sim}{r}(t) = (\cos t)\underset{\sim}{i} + (\sin t)\underset{\sim}{j} + (t)\underset{\sim}{k}$

(B) $\underset{\sim}{r}(t) = (t\cos t)\underset{\sim}{i} + (t\sin t)\underset{\sim}{j} + (t)\underset{\sim}{k}$

(C) $\underset{\sim}{r}(t) = (t\cos t)\underset{\sim}{i} + (t\sin t)\underset{\sim}{j} + \left(\frac{1}{t}\right)\underset{\sim}{k}$

(D) $\underset{\sim}{r}(t) = (\cos t)\underset{\sim}{i} + (\sin t)\underset{\sim}{j} + \left(\frac{1}{t}\right)\underset{\sim}{k}$

10. A particle is moving along a straight line. At time t, its velocity is v and its displacement from a fixed origin is x.

If $\frac{dv}{dx} = \frac{1}{2v}$ which of the following best describes the particle's acceleration and velocity?

(A) Constant acceleration and constant velocity

(B) Constant acceleration and decreasing velocity

(C) Constant acceleration and increasing velocity

(D) Increasing acceleration and increasing velocity

End of Section I

Section II (90 marks)

Attempt Questions 11 to 15
Allow about 2 hour 45 minutes for this section

Answer each question in the appropriate writing page. Extra writing pages are available.
All necessary working should be shown in every question.

Question 11 (18 marks)

(a) By first writing $\sqrt{3}+i$ and $\sqrt{3}-i$ in exponential form, **2**

express $\left(\sqrt{3}+i\right)^{12}+\left(\sqrt{3}-i\right)^{12}$ in the form of a^b where a and b are integers.

(b) A polygonal number is an integer which can be represented as a series of dots arranged in the shape of a regular polygon. Triangular numbers, square numbers and pentagonal numbers are examples of polygonal numbers.

For an r-sided regular polygon, where $r \in \mathbb{Z}^+, r \geq 3$, the nth polygonal number $P_r(n)$ is given by

$$P_r(n) = \frac{(r-2)n^2-(r-4)n}{2}$$

where $n \in \mathbb{Z}^+$. Hence, the nth triangular number can be expressed as $P_3(n) = \frac{n^2+n}{2}$.

(i) The nth pentagonal number can be represented by the arithmetic series **1**

$$P_5(n) = 1+4+7+\cdots+(3n-2)$$

Hence show that $P_5(n) = \frac{3n^2-n}{2}$ for $n \in \mathbb{Z}^+$.

(ii) The nth polygonal number, $P_r(n)$, can be represented by the series **3**

$$\sum_{m=1}^{n}\left(1+(m-1)(r-2)\right)$$

where $r \in \mathbb{Z}^+, r \geq 3$.

Use mathematical induction to prove that

$$P_r(n) = \frac{(r-2)n^2-(r-4)n}{2}$$

where $n \in \mathbb{Z}^+$.

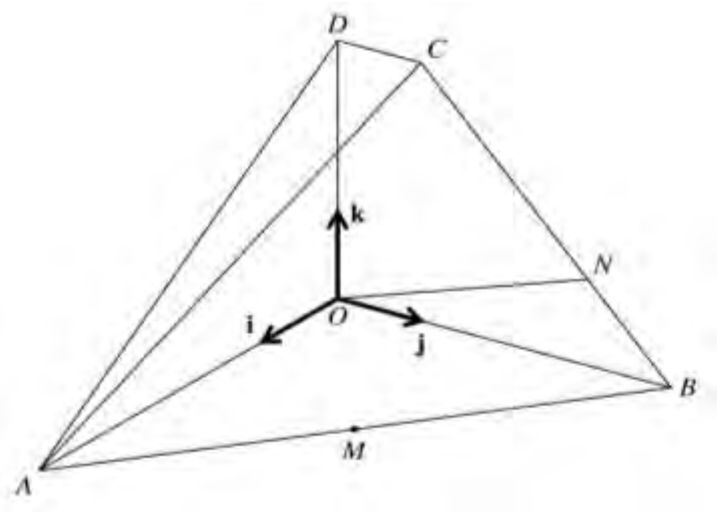
Question 11 continues on the next page

(c) Given $a \in Z$, prove that if $3a^2 - 4a + 5$ is even, then a is odd. **2**

(d) Find the equations of a sphere whose centre is at $(1,0,1)$ and touches the line **3**

$\underset{\sim}{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ where $\lambda \in \mathbb{R}$.

(e) In the diagram below, $OABCD$ is a solid figure where $|\overrightarrow{OA}| = |\overrightarrow{OB}| = 4$ units and $|\overrightarrow{OD}| = 3$ units. The edge $\overrightarrow{OD}$ is vertical, $\overrightarrow{DC}$ is parallel to $\overrightarrow{OB}$ and $|\overrightarrow{DC}| = 1$ unit. The base, OAB, is horizontal and $\angle AOB = 90°$. Unit vectors $\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$ are parallel to $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OD}$ respectively. The midpoint of $\overrightarrow{AB}$ is M and the point N on $\overrightarrow{BC}$ is such that $\overrightarrow{NC} = 2\overrightarrow{BN}$.



(i) Express vectors $\overrightarrow{MD}$ and $\overrightarrow{ON}$ in terms of $\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$. **2**

(ii) Calculate the angle between $\overrightarrow{MD}$ and $\overrightarrow{ON}$. **2**

(iii) Using vector methods, show that the length of the perpendicular from M to $\overrightarrow{ON}$ is $\sqrt{\frac{22}{5}}$ units. **3**

End of Question 11

Question 12 (18 marks)

(a) Let α be a real number and suppose that z is a complex number such that

$$z + \frac{1}{z} = 2\cos\alpha$$

You may assume that $z^n + \frac{1}{z^n} = 2\cos n\alpha$ for all positive integer n.

Let $\omega = z + \frac{1}{z}$.

(i) Show that $\omega^4 + \omega^3 - 3\omega^2 - 2\omega = z + \frac{1}{z} + z^2 + \frac{1}{z^2} + z^3 + \frac{1}{z^3} + z^4 + \frac{1}{z^4}$. **2**

(ii) Find the ninth roots of unity. **2**

(iii) Hence or otherwise, find all solution of **2**

$16(\cos\alpha)^4 + 8(\cos\alpha)^3 - 12(\cos\alpha)^2 - 4\cos\alpha + 1 = 0$.

(b)

(i) Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x} = \ln 2$. **3**

(ii) By making the substitution $u = a - x$, show that **1**

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

(iii) Hence or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}\,dx$. **2**

(c) Let $I_n = \int_{e^{-1}}^{1} (1 + \log_e x)^n\,dx$ and $J_n = \int_{e^{-1}}^{1} (\log_e x)(1 + \log_e x)^n\,dx$

for $n = 0, 1, 2, 3$ ...

(i) Show that $I_n = 1 - nI_{n-1}$ for $n = 1, 2, 3$ ... **2**

(ii) Show that $J_n = 1 - (n+2)I_n$ for $n = 0, 1, 2\ 3$ ... **2**

(iii) Hence find the value of J_3 in simplest exact form. **2**

End of Question 12

Question 13 (18 marks)

(a) (i) Use De Moivre's Theorem to solve the equation $z^3 = 4\sqrt{2}(1+i)$. **2**

(ii) By considering the roots of $z^3 = 4\sqrt{2}(1+i)$, show that $\cos\frac{7\pi}{12} + \cos\frac{\pi}{12} = \cos\frac{\pi}{4}$. **2**

(b) Sketch the intersection of the following. **3**

$$|z-3| = 3 \text{ and } -\frac{\pi}{4} \le Arg(z) \le \frac{\pi}{4}$$

(c) (i) Find real numbers a, b and c such that $\frac{10}{(x+1)(x^2+4)} \equiv \frac{a}{x+1} + \frac{bx+c}{x^2+4}$ **3**

(ii) Hence find $\int \frac{10}{(x+1)(x^2+4)}dx$. **2**

(d) Consider the line ℓ_1 joining $\begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$.

(i) Determine the vector equation of ℓ_1. **2**

Another line, ℓ_2, is defined by the vector equation $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$, where $\lambda, a \in \mathbb{R}$.

(ii) Find the possible values of a when the angle between ℓ_1 and ℓ_2 is $\frac{\pi}{4}$. **2**

(iii) ℓ_1 and ℓ_2 have a unique point of intersection when $a \neq 2$. Find the point of intersection in terms of a. **2**

End of Question 13

Question 14 (18 marks)

(a) Prove by contradiction that there are no rational solutions to the equation $x^3 + 3x + 3 = 0$. **4**

(b) Given a, b, c are positive real numbers.

(i) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$ **2**

(ii) Hence or otherwise, prove that $a^3 + b^3 + c^3 \geq 3abc$ **1**

(iii) Hence or otherwise, prove that $(1 + a^3)(1 + b^3)(1 + c^3) \geq \left(\frac{ab+bc+ca+1}{2}\right)^3$ **3**

(c) The only force acting on a particle moving in a straight line is a resistance $m\lambda(c + v)$ acting in the same line. The mass of the particle is m, its velocity is v, and λ and c are positive constants. The particle starts to move to with velocity $u > 0$ and comes to rest after T seconds. After half the time has elapsed, the particle's velocity is a quarter of its initial velocity. **4**

Show that

$$c = \frac{u}{8}$$

(d) A particle is moving in simple harmonic motion with centre around the origin, starting at $x = m$, where $m > 0$. The displacement equation is given by $x = a\cos(nt + \alpha)$. After 1 second, the particle is at $x = r$, where $r > m$ and after another second, it returns to $x = r$. **4**

Show that

$$\cos n = \frac{r+m}{2r}$$

End of Question 14

Question 15 (18 marks)

(a) A particle of unit mass is moving in horizontal motion, subject to a resistance force of $v^2 + v^3$, where v is the object's velocity. The particle has initial velocity v_0, where $v_0 > 0$.

(i) Find the distance s travelled by the particle when its velocity is $\frac{v_0}{2}$. **3**

(ii) Show that the time T taken to travel the distance s is $T = \frac{1}{v_0} - s$. **3**

(iii) Show that if the particle starts at the origin, then **2**

$$v = \frac{v_0}{v_0 x + v_0 t + 1}$$

satisfies the equation of motion.

(b) A particle of unit mass is thrown vertically downwards with an initial velocity of v_0. It experiences a resistive force of magnitude kv^2 where v is its velocity. Taking downwards as the positive direction, the equation of motion of the particle is given by $\ddot{x} = g - kv^2$. Let V be the terminal velocity of the particle.

(i) Explain why $V = \sqrt{\frac{g}{k}}$. **1**

(ii) Show that $v^2 = V^2 + (v_0^2 - V^2)e^{-2kx}$. **4**

(c) z_1 and z_2 are two complex numbers representing the two points A and B in the Argand diagram. z_3 is a complex number representing the point C such that $|AB|:|AC| = 1:4$. z_4 is a complex number representing the point D, such that $|OB|:|OD| = 1:k$, for some constant k and O is the origin. The points A, B and C are collinear.

(i) Find z_3 in terms of z_1 and z_2. **2**

(ii) Given that $z_2 - z_1$ and $z_4 - z_3$ are perpendicular, prove that **3**

$$k = \frac{4|z_2|^2 - 7|z_1||z_2|\cos\theta + 3|z_1|^2}{|z_2|^2 - |z_1||z_2|\cos\theta},$$

where θ is the angle between z_1 and z_2.

End of paper

2023 Yr 12 Ext 2 Trial

1. (D)

2. $(-3+3i)^3 = (-3)^3 + 3(-3)^2(3i) + 3(-3)(3i)^2 + (3i)^3$

$= -27 + 81i - 81i^2 + 27i^3$

$= -27 + 81i + 81 - 27i$

$= 54 + 54i$

(A)

3. $\begin{pmatrix} 3 \\ -6 \\ -15 \end{pmatrix} = -3 \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$

(B)

4. $\int \frac{dx}{\sqrt{7-6x-x^2}} = \int \frac{dx}{\sqrt{-(x^2+6x+9)+9+7}}$

$= \int \frac{dx}{\sqrt{16-(x+3)^2}}$

$= \sin^{-1} \frac{x+3}{4} + C$

(D)

5. (D)

6. $P(z) = a(\bar{z}+\alpha)(z-\alpha)(z+\beta)(z+\beta)$

(C)

7. $\int 3\sqrt{x} \ln x \, dx$

$u = \ln x \qquad v = \frac{2}{3}x^{\frac{3}{2}}$

$u' = \frac{1}{x} \qquad v' = \sqrt{x}$

$3\left[\frac{2}{3}x^{\frac{3}{2}} \ln x - \int \frac{2}{3}x^{\frac{3}{2}} \times \frac{1}{x}\, dx\right]$

$= 2x^{\frac{3}{2}} \ln x - 2\int \sqrt{x}\, dx$

$= 2x^{\frac{3}{2}} \ln x - 2 \times \frac{2}{3}x^{\frac{3}{2}} + C$

$= 2x^{\frac{3}{2}}\left(\ln x - \frac{2}{3}\right) + C$

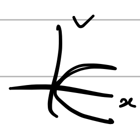
(A)

8. $v^2 = b^2\left[\frac{a^2}{b^2} - x^2\right]$

$n = b$

$T = \frac{2\pi}{b}$

(D)



$\int 2v\,dv = \int dx$

$\frac{2v^2}{2} = x + C \therefore v^2 = x.$

$\frac{dv}{dx} = \frac{1}{2v} \qquad v\frac{dv}{dx} = \frac{1}{2}$

$a = \frac{1}{2}.$

9. (B)

10. (C)

MATHEMATICS EXT 2: Question 11

Suggested Solutions | **Marks** | **Marker's Comments**

a, $\sqrt{3}+i = 2e^{i\pi/6}$

$\sqrt{3}-i = 2e^{-i\pi/6}$ — 1

$(\sqrt{3}+i)^{12} + (\sqrt{3}-i)^{12} = (2e^{i\pi/6})^{12} + (2e^{-i\pi/6})^{12}$

$= 2^{12}e^{i2\pi} + 2^{12}e^{-i2\pi}$

$= 2^{12} + 2^{12}$

$= 2^{13}$ — 1

bi, $P_5(n) = 1+4+7+\dots+(3n-2)$

This is an A.P. with $a=1$, $l=3n-2$ and n terms

Must use sum of an A.P. formula.

$\therefore P_5(n) = \frac{n}{2}(1+3n-2)$

$= \frac{n}{2}(3n-1)$

$= \frac{3n^2-n}{2}$, as required

ii, Base case: $n=1$

Inducting on either $n \geq 1$ or $r \geq 3$ was accepted.

$LHS = P_r(1) = \sum_{m=1}^{1}(1+(m-1)(r-2))$, by defn

$= 1+0(r-2)$

$= 1$

For completeness, this should have been a "double induction" on both variables.

$RHS = \frac{(r-2)\times 1^2 - (r-4)\times 1}{2}$

$= \frac{2}{2}$

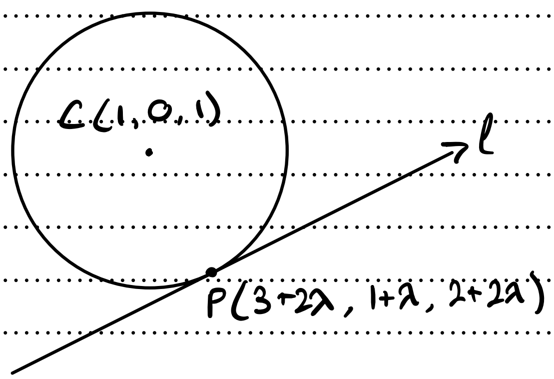
$= 1 = LHS$ — Base Case 1

$\therefore$ base case is true

MATHEMATICS EXT 2: Question 10

Suggested Solutions	Marks	Marker's Comments
Assume for some $k \geq 1$, $k \in \mathbb{Z}$ that		
$P_r(k) = \dfrac{(r-2)k^2 - (r-4)k}{2}$ (*)		
Now we wish to prove that		
$P_r(k+1) = \dfrac{(r-2)(k+1)^2 - (r-4)(k+1)}{2}$		
$LHS = P_r(k+1)$		
$= \sum_{m=1}^{k+1} \left(1 + (m-1)(r-2)\right)$		
$= \sum_{m=1}^{k} \left(1 + (m-1)(r-2)\right) + [1 + k(r-2)]$		
$= P_r(k) + 1 + k(r-2)$		
$= \dfrac{(r-2)k^2 - (r-4)k}{2} + 1 + k(r-2)$	1 using assumption	
by (*)		
$= \dfrac{(r-2)k^2 - (r-4)k + 2 + 2k(r-2)}{2}$		
$= \dfrac{(r-2)k^2 + 2k(r-2) + (r-2) - (r-2) + 2 - (r-4)k}{2}$		
$= \dfrac{(r-2)(k^2+2k+1) - r + 4 - (r-4)k}{2}$		
$= \dfrac{(r-2)(k+1)^2 - (r-4)(k+1)}{2}$		
$= RHS$		
$\therefore$ statement is true for $n = k+1$ if it is true for $n = k$.	1	Getting to the end w/o skipping important algebra.
$\therefore$ The statement is true by Mathematical induction for integers $n \geq 1$.		

MATHEMATICS EXT 2: Question 10

Suggested Solutions	Marks	Marker's Comments
c) Consider the equivalent contrapositive: If a is even, then $3a^2-4a+5$ is odd.		Students used: • Direct proof • Contrapositive • Contradiction
Let $a=2k$, $k\in\mathbb{Z}$	1 progress	All are valid if completed correctly
$3a^2-4a+5 = 3(2k)^2-4(2k)+5$ $= 2(6k^2-4k+2)+1$ $= 2M+1$, where $M=6k^2-4k+2$ is an integer		
$\therefore 3a^2-4a+5$ is odd $\therefore$ original statement is true, since the equivalent contrapositive is true.	1 complete proof	
d) 		Again, multiple ways of approaching this question. Only one is shown.
Let $C=(1,0,1)$ be the centre of the sphere and P the point of tangency between the sphere and line. Then since P lies on the line, it has coordinates $(3+2\lambda, 1+\lambda, 2+2\lambda)$ for some λ.		
Now, we want $\overrightarrow{CP}\cdot \underset{\sim}{v}=0$, where $\underset{\sim}{v}$ is the direction of ℓ		

MATHEMATICS EXT 2: Question 10

Suggested Solutions	Marks	Marker's Comments
$\therefore \begin{pmatrix} 2+2\lambda \\ 1+\lambda \\ 1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0$	1 recognising radius $\perp$ tangent	
$4+4\lambda+1+\lambda+2+4\lambda=0$		
$9\lambda+7=0$		
$\lambda = \frac{-7}{9}$	1 valid progress	
$\therefore$ Equation of sphere is		
$\left\| \underset{\sim}{r} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\| = \|\overrightarrow{CP}\|$		
$= \left\| \begin{pmatrix} 4/9 \\ 2/9 \\ -5/9 \end{pmatrix} \right\|$		
$\left\| \underset{\sim}{r} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\| = \frac{\sqrt{5}}{3}$	1 final	
or		
$(x-1)^2+y^2+(z-1)^2 = \frac{5}{9}$		

11e) i) Given $\vec{OA} = 4\underset{\sim}{i}$, $\vec{OB} = 4\underset{\sim}{j}$, $\vec{OC} = 3\underset{\sim}{k}$, $\vec{DC} = \underset{\sim}{j}$

$$\vec{MD} = \vec{MB} + \vec{BO} + \vec{OD}$$
$$= \tfrac{1}{2}\vec{AB} + \vec{BO} + \vec{OD}$$
$$= \tfrac{1}{2}(4\underset{\sim}{j} - 4\underset{\sim}{i}) - 4\underset{\sim}{j} + 3\underset{\sim}{k}$$
$$= 2\underset{\sim}{j} - 2\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}$$
$$= -2\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$$

Better written in this order. 1

$$\vec{ON} = \vec{OB} + \vec{BN}$$
$$= \vec{OB} + \tfrac{1}{3}\vec{BC}$$
$$= 4\underset{\sim}{j} + \tfrac{1}{3}\left((3\underset{\sim}{k} + \underset{\sim}{j}) - 4\underset{\sim}{j}\right)$$
$$= 4\underset{\sim}{j} + \underset{\sim}{k} - \underset{\sim}{j}$$
$$= 3\underset{\sim}{j} + \underset{\sim}{k}$$ 1

ii) Let the required angle be θ.
Then

$$\vec{MD} \cdot \vec{ON} = |MD||ON|\cos\theta$$ 1

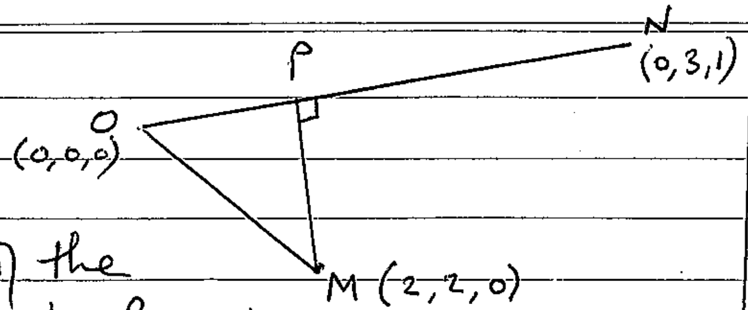
$$\cos\theta = \left|\frac{\vec{MD} \cdot \vec{ON}}{|\vec{MD}||\vec{ON}|}\right|$$

abs. value to make angle acute,

$$= \left|\frac{0 - 6 + 3}{\sqrt{17}\sqrt{10}}\right|$$
$$= \frac{3}{\sqrt{170}}$$

$\Rightarrow$ $\theta = 76°42'$ to nearest minute 1

$76{\cdot}7°$, $103°18'$, $103{\cdot}3°$ also accepted as question was not very discerning iii)



Let point of intersection of the perpendicular be P.

Method 1 Find OP, the projection of OM on to ON, Then use Pythagoras on ΔOPM.

$$OP = \mathrm{Proj}_{\vec{ON}}\vec{OM} = \frac{\vec{OM}\cdot\vec{ON}}{|\vec{ON}|^2}\vec{ON}$$ 1 for correct projection formula

$$= \frac{0+6+0}{3^2+1^2}\vec{ON}$$

$$= \frac{6}{10}\vec{ON}$$

$$\therefore |\vec{OP}| = \frac{3}{5}|\vec{ON}|$$

$$= \frac{3\sqrt{10}}{5}.$$ 1 for a correct formulation for answer

Now $|PM|^2 = OM^2 - OP^2$ (Pythagoras)

$$= 8 - 90/25$$

$$= 110/25$$

$$= 22/5$$

$$\therefore |PM| = \sqrt{\frac{22}{5}} \text{ units.}$$ 1 for correct answer, suitably presented (ie, enough working)

(N.B. The diagram was useful to see what was required.

iii) Method 2 Find the co-ordinates of P then calculate the length of the resulting vector MP.

$$\vec{OP} = \lambda \vec{ON} \text{ for some } \lambda \in \mathbb{R}$$
$$= 3\lambda \underset{\sim}{j} + \lambda \underset{\sim}{k}$$

Now $\vec{OP} \cdot \vec{PM} = 0$ (Perpendicular)

i.e. $(3\lambda \underset{\sim}{j} + \lambda \underset{\sim}{k}) \cdot (2\underset{\sim}{i} + (2-3\lambda)\underset{\sim}{j} - \lambda \underset{\sim}{k}) = 0$

$$\therefore 3\lambda(2-3\lambda) - \lambda^2 = 0$$
$$6\lambda - 9\lambda^2 - \lambda^2 = 0$$
$$10\lambda^2 = 6\lambda$$
$$\lambda = 3/5 \quad (\lambda \neq 0)$$

1 for $\lambda = 3/5$ or equivalent

$$\therefore \vec{OP} = \frac{9}{5}\underset{\sim}{j} + \frac{3}{5}\underset{\sim}{k}$$

$$\therefore \vec{PM} = 2\underline{i} + \frac{1}{5}\underline{j} - \frac{3}{5}\underline{k}$$

$$\therefore |PM| = \sqrt{4 + \frac{1}{25} + \frac{9}{25}} \qquad A$$

1 for a form of $|PM|$

$$= \sqrt{\frac{100+1+9}{25}} \qquad B$$

$$= \sqrt{\frac{110}{25}} \qquad C$$

$$= \sqrt{\frac{22}{5}} \text{ units.} \qquad D$$

1 for correctly deduced answer. (It is NOT satisfactory to jump from A to D.)

MATHEMATICS Extension 2: Question 12

Suggested Solutions	Marks	Marker's Comments
a) i) RTP: $\omega^4 + \omega^3 - 3\omega^2 - 2\omega = z + \frac{1}{z} + z^2 + \frac{1}{z^2} + z^3 + \frac{1}{z^3} + z^4 + \frac{1}{z^4}$		
$LHS = (z+\frac{1}{z})^4 + (z+\frac{1}{z})^3 - 3(z+\frac{1}{z})^2 - 2(z+\frac{1}{z})$		
$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4} + z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} - 3(z^2 + 2 + \frac{1}{z^2}) - 2z - \frac{2}{z}$		1 expanding
$= z^4 + \frac{1}{z^4} + z^3 + \frac{1}{z^3} + 4z^2 - 3z^2 + \frac{4}{z^2} - \frac{3}{z^2} + 3z - 2z + \frac{3}{z} - \frac{2}{z} + 6 - 6$		1 working
$= z^4 + \frac{1}{z^4} + z^3 + \frac{1}{z^3} + z^2 + \frac{1}{z^2} + z + \frac{1}{z}$		
$= RHS$		
ii) $z^9 = 1$		
$z^9 = \cos 0 + i\sin 0$		
let $z = r(\cos\theta + i\sin\theta)$		
$\therefore r^9(\cos 9\theta + i\sin 9\theta) = \cos 0 + i\sin 0$ (De Moivre's)		
$\therefore r = 1$		
$9\theta = 0 + 2k\pi, \quad k \in \mathbb{Z}$	1	must mention that k is an integer.
$\theta = \frac{2k\pi}{9}$		
$\therefore$ roots are:		
$1, \quad \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}, \quad \cos(\frac{2\pi}{9}) + i\sin(-\frac{2\pi}{9})$		
$\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}, \quad \cos(-\frac{4\pi}{9}) + i\sin(-\frac{4\pi}{9})$		1
$\cos\frac{6\pi}{9} + i\sin\frac{6\pi}{9}, \quad \cos(-\frac{6\pi}{9}) + i\sin(-\frac{6\pi}{9})$		
$\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}, \quad \cos(-\frac{8\pi}{9}) + i\sin(-\frac{8\pi}{9})$		

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments

iii) $16(\cos\alpha)^4 + 8(\cos\alpha)^3 - 12(\cos\alpha)^2 - 4\cos\alpha + 1 = 0$

$(2\cos\alpha)^4 + (2\cos\alpha)^3 - 3(2\cos\alpha)^2 - 2(2\cos\alpha) + 1 = 0$

$\left(z+\frac{1}{z}\right)^4 + \left(z+\frac{1}{z}\right)^3 - 3\left(z+\frac{1}{z}\right)^2 - 2\left(z+\frac{1}{z}\right) + 1 = 0$

$w^4 + w^3 - 3w^2 - 2w + 1 = 0$

$z^4 + \frac{1}{z^4} + z^3 + \frac{1}{z^3} + z^2 + \frac{1}{z^2} + z + \frac{1}{z} + 1 = 0$

$z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ — 1

for $z^9 = 1$

$z^9 - 1 = 0$

$(z-1)(z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$

$\therefore$ the solutions for $z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ are the complex solutions for $z^9 = 1$

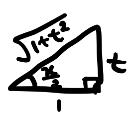
$\therefore$ solutions are: $\pm\frac{2\pi}{9}, \pm\frac{4\pi}{9}, \pm\frac{6\pi}{9}, \pm\frac{8\pi}{9}$ — 1

b) i) let $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\frac{dx}{dt} = \frac{2}{\sec^2\frac{x}{2}}$

$= 2\cos^2\frac{x}{2}$



$= \frac{2}{1+t^2}$

$\therefore dx = \frac{2\,dt}{1+t^2}$

when $x = \frac{\pi}{2}$, $t = 1$

$x = 0$, $t = 0$

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x} = \int_0^1 \frac{\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}$	1	
$= \int_0^1 \frac{2dt}{1+t^2+1-t^2+2t}$		
$= \int_0^1 \frac{2dt}{2+2t}$		
$= \int_0^1 \frac{dt}{1+t}$		
$= \left[\ln\|1+t\|\right]_0^1$	1	
$= \ln\|1+1\| - \ln\|1+0\|$		Show substitution.
$= \ln 2 - 0$	1	
$= \ln 2$		
ii) $\int_0^a f(x)\,dx$		
let $u = a-x$, $x = a-u$		
$\frac{du}{dx} = -1$		
$dx = -du$		
when $x=a$, $u=0$		
$x=0$, $u=a$		
$\therefore \int_0^a f(x)\,dx = \int_a^0 f(a-u)(-du)$		
$= \int_0^a f(a-u)\,du$	1	
as u is a dummy variable		
$= \int_0^a f(a-x)\,dx$		
iii) $\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}\,dx$		
$= \int_0^{\frac{\pi}{2}} \frac{(\frac{\pi}{2}-x)}{1+\cos(\frac{\pi}{2}-x)+\sin(\frac{\pi}{2}-x)}\,dx$ from (ii)		

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
$= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{1 + \sin x + \cos x}\,dx$	1	
$= \int_0^{\frac{\pi}{2}} \left(\frac{\frac{\pi}{2}}{1 + \cos x + \sin x} - \frac{x}{1 + \cos x + \sin x} \right) dx$		
$\therefore 2 \int_0^{\frac{\pi}{2}} \frac{x}{1 + \cos x + \sin x}\,dx$		
$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos x + \sin x}\,dx$		
$= \frac{\pi}{2} (\ln 2)$ from (i)		
$\therefore \int_0^{\frac{\pi}{2}} \frac{x}{1 + \cos x + \sin x}\,dx = \frac{\pi}{4} \ln 2$	1	
c) i) $I_n = \int_{e^{-1}}^{1} (1 + \log_e x)^n\,dx$		
$u = (1 + \ln x)^n \qquad v' = 1$		
$u' = \frac{n}{x}(1 + \ln x)^{n-1} \qquad v = x$		
$I_n = \left[x(1 + \ln x)^n \right]_{e^{-1}}^{1} - \int_{e^{-1}}^{1} x \left(\frac{n}{x}\right)(1 + \ln x)^{n-1}\,dx$	1	
$= (1 + \ln(1))^n - e^{-1}(1 + \ln e^{-1})^n - n \int_{e^{-1}}^{1} (1 + \ln x)^{n-1}\,dx$		
$= 1 - e^{-1}(1 - 1)^n - n\, I_{n-1}$	1	
$= 1 - n\, I_{n-1}$		

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
ii) $J_n = \int_{e^{-1}}^{1} (\ln x)(1+\ln x)^n \, dx$		
$= \int_{e^{-1}}^{1} (1+\ln x - 1)(1+\ln x)^n \, dx$	1	
$= \int_{e^{-1}}^{1} \left[(1+\ln x)^{n+1} - (1+\ln x)^n\right] dx$		
$= \int_{e^{-1}}^{1} (1+\ln x)^{n+1} \, dx - \int_{e^{-1}}^{1} (1+\ln x)^n \, dx$		
$= I_{n+1} - I_n$	1	
$= 1-(n+1)I_n - I_n$		
$= 1-(n+2)I_n$		
iii) $J_3 = 1-(3+2)I_3$		
$= 1 - 5(1-3I_2)$		
$= 1 - 5 + 15(1-2I_1)$		
$= -4+15-30(1-1I_0)$		
$= 11 - 30 + 30\int_{e^{-1}}^{1} 1 \, dx$	1	showing $I_0 = 1-e^{-1}$
$= -19 + 30[x]_{e^{-1}}^{1}$		
$= -19 + 30(1-e^{-1})$		
$= 11 - 30e^{-1}$	1	

Student Number ____________________

Question 13

a) i) $z^3 = 4\sqrt{2}(1+i)$

$= 8\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Let $z = r(\cos\theta + i\sin\theta)$

then $z^3 = r^3(\cos 3\theta + i\sin 3\theta)$

1 mark for using De Moivre's Theorem correctly

$\therefore r^3(\cos 3\theta + i\sin 3\theta) = 8\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$r^3 = 8 \qquad 3\theta = \frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}$

allow

$r = 2 \qquad \theta = \frac{\pi + 8k\pi}{12}$

$\therefore$ when $k=0$, $z = 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$k=1$, $z = 2\left(\cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}\right)$

$= 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$k=2$, $z = 2\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$

$= 2\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$

1 mark for all 3 solutions correct

ii) $\sum\alpha = 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) + 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) + 2\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right) = 0$

* Equating real parts : $2\cos\frac{\pi}{12} + 2\cos\frac{3\pi}{4} + 2\cos\left(-\frac{7\pi}{12}\right) = 0$

or collect parts

$\cos\frac{\pi}{12} - \cos\frac{\pi}{4} + \cos\frac{7\pi}{12} = 0$

* cis: not allowed unless defined

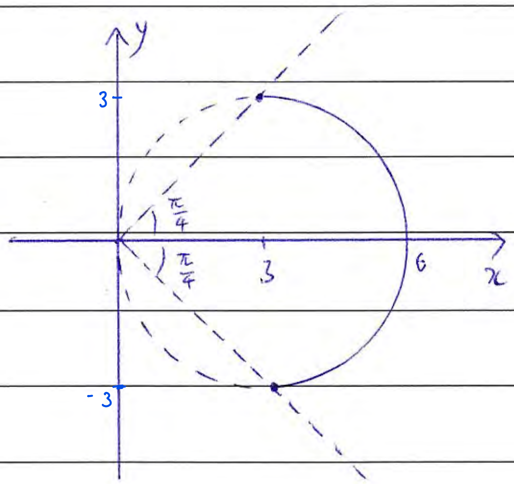
$\therefore \cos\frac{\pi}{12} + \cos\frac{7\pi}{12} = \cos\frac{\pi}{4}$

* e.c. allowed only if the error did not omit skills that needed to be assessed

1 mark for sum of roots

1 mark for equating real parts and working through logically to get to the required result

b)



1 mark for circle + centre, radius

1 mark for lines + angles

1 mark for correct solution

Student Number ____________________

c, i, $\dfrac{10}{(x+1)(x^2+4)} \equiv \dfrac{a}{x+1} + \dfrac{bx+c}{x^2+4}$

$10 = a(x^2+4) + (bx+c)(x+1)$

1 mark for setting this statement

$x=-1 \Rightarrow 10 = 5a \quad \therefore a = 2$

1 mark for 1 correct answer

$x = 0 \Rightarrow 10 = 4a + c \quad \therefore c = 2$

1 mark for all correct answers

$x = 1 \Rightarrow 10 = 5a + 2(b+c) \quad \therefore b = -2$

$\therefore a = 2, \quad b = -2, \quad c = 2$

ii, $\displaystyle\int \frac{10}{(x+1)(x^2+4)}\,dx$

$= \displaystyle\int \frac{2}{x+1} + \frac{-2x+2}{x^2+4}\,dx$

1 mark for getting the first integral correct or equivalent

1 mark for correct solution

$= \displaystyle\int \frac{2}{x+1} - \frac{2x}{x^2+4} + \frac{2}{x^2+4}\,dx$

$= 2\ln|x+1| - \ln|x^2+4| + \tan^{-1}\frac{x}{2} + C$

missing absolute value: allowed

missing C: allowed

d, i, $\begin{pmatrix}-1\\0\\3\end{pmatrix} - \begin{pmatrix}-3\\-1\\4\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$

$\therefore l_1 : \underset{\sim}{r} = \begin{pmatrix}-1\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}, \quad \underline{\lambda \in \mathbb{R}}$ ☆

1 mark for correctly finding the direction vector

OR

1 mark for correct solution, defining lambda

$\underset{\sim}{r} = \begin{pmatrix}-3\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}, \quad \lambda \in \mathbb{R}$

Student Number ____________

ii) $l_2 : \underset{\sim}{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}, \quad \lambda, a \in \mathbb{R}$

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix} = \left|\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right| \left|\begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}\right| \cos\theta$$

1 mark for setting this statement

1 mark for the correct solution

<u>Case I</u> $\theta = \frac{\pi}{4}$

$2a + 1 + 1 = \sqrt{6} \cdot \sqrt{a^2+2} \cdot \frac{1}{\sqrt{2}}$

$2a + 2 = \sqrt{3(a^2+2)}$

$(2a+2)^2 = 3(a^2+2)$

$a^2 + 8a - 2 = 0$

$\therefore a = -4 + 3\sqrt{2}$

(reject negative)

<u>Case II</u> $\theta = \frac{3\pi}{4}$

$2a + 1 + 1 = \sqrt{6} \cdot \sqrt{a^2+2} \cdot \left(-\frac{1}{\sqrt{2}}\right)$

$2a + 2 = -\sqrt{3(a^2+2)}$

$(2a+2)^2 = 3(a^2+2)$

$a^2 + 8a - 2 = 0$

$\therefore a = -4 - 3\sqrt{2}$

(reject positive)

$$\therefore a = -4 \pm 3\sqrt{2}$$

iii) $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$

$$\begin{cases} -1 + 2\lambda = a\mu & (1) \\ \lambda = 1 + \mu & (2) \\ 3 - \lambda = 2 - \mu & (3) \end{cases}$$

1 mark for setting equations 1 and 2 OR equations 1 and 3

1 mark for the correct solution

Sub (2) into (1) : $-1 + 2(1+\mu) = a\mu$

$1 + 2\mu = a\mu$

$\therefore \mu = \frac{1}{a-2} \quad (a \neq 2)$

$\therefore$ The point of intersection $= \left(\frac{a}{a-2},\ 1 + \frac{1}{a-2},\ 2 - \frac{1}{a-2} \right)$

$= \left(\frac{a}{a-2},\ \frac{a-1}{a-2},\ \frac{2a-5}{a-2} \right)$ ✓

Question 14

ⓐ Prove by contradiction that there are no rational solutions to the equation $x^3+3x+3=0$

Solutions: Assume $x^3+3x+3=0$ has a rational solution

i.e $\frac{p}{q}$ is a solution and the $HCF(p,q)=1$ (1 mark)

$$\left(\frac{p}{q}\right)^3+3\left(\frac{p}{q}\right)+3=0$$

$$p^3+3pq^2+3q^3=0$$ (1 mark)

Method 1:

Case 1: p and q are odd

$\therefore$ p^3 is odd

$3q^3$ is odd

$3pq^2$ is odd

odd + odd + odd $\neq$ even

$\therefore$ $p^3+3pq^2+3q^3\neq 0$ if both p and q are odd

Case 2: p is odd and q is even

$\therefore$ p^3 is odd

$3q^3$ is even

$3pq^2$ is even

odd + even + even $\neq$ even

$\therefore$ $p^3+3pq^2+3q^2\neq 0$ if p is odd and q is even

1 mark for considering all cases

Case 3: p is even and q is odd

$\therefore$ p^3 is even

$3pq^2$ is even

$3q^3$ is odd

even + even + odd $\neq$ even

$\therefore$ $p^3+3pq^2+3q^2\neq 0$ if p is even and q is odd

1 mark for showing that each case leads to contradiction

Case 4: p and q can not be both even since $HCF(p,q)=1$

$\therefore$ $p^3+3pq^2+3q^3\neq 0$ contradicts the assumption.

Method 2: $p^3 = -3(pq^2 + q^3)$

$= -3q^2(p+q)$

p^3 is divisible by 3

$\therefore$ p is divisible by 3

$\therefore$ $\exists k \in \mathbb{Z},\ p = 3k$

1 mark for considering all cases

$(3k)^3 = -3q^2(p+q)$

$-9k^3 = q^2(p+q)$

q^2 is divisible 9 $\therefore$ q is divisible by 3

but $HCF(p,q) = 1$

OR $p+q$ is divisible by 9

$\exists n \in \mathbb{Z},\ p+q = 9n$

$q = 9n - p$

$= 9n - 3k$

$= 3(3n-k)$ and q is divisible by 3

1 mark for proving that it leads to contradiction

Contradiction

(b)

(i) $(a-b)^2 \geq 0$ $\therefore$ $a^2 + b^2 \geq 2ab$ ✓

Similarly $a^2 + c^2 \geq 2ac$

$b^2 + c^2 \geq 2bc$

$2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$

$\therefore$ $a^2+b^2+c^2 \geq ab+bc+ca$ ✓

(ii) $(a+b+c)(a^2+b^2+c^2) \geq (a+b+c)(ab+bc+ca)$

Expand both side and simplify, we obtain ✓

$a^3 + b^3 + c^3 \geq 3abc$

(iii) $\dfrac{a^3}{1+a^3} + \dfrac{b^3}{1+b^3} + \dfrac{1}{1+c^3} \geq \dfrac{3ab}{\sqrt[3]{(1+a^3)(1+b^3)(1+c^3)}}$ ① ✓

$\dfrac{a^3}{1+a^3} + \dfrac{1}{1+b^3} + \dfrac{c^3}{1+c^3} \geq \dfrac{3ac}{\sqrt[3]{(1+a^3)(1+b^3)(1+c^3)}}$ ②

$\dfrac{1}{1+a^3} + \dfrac{b^3}{1+b^3} + \dfrac{c^3}{1+c^3} \geq \dfrac{3bc}{\sqrt[3]{(1+a^3)(1+b^3)(1+c^3)}}$ ③

$$\frac{1}{1+a^3}+\frac{1}{1+b^3}+\frac{1}{1+c^3} \geq \frac{3}{\sqrt[3]{(1+a^3)(1+b^3)(1+c^3)}} \quad (4)$$ ✓

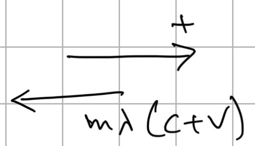
(1) + (2) + (3) + (4) $\therefore$

$$6 \geq \frac{3(ab+bc+ca+1)}{\sqrt[3]{(1+a^3)(1+b^3)(1+c^3)}}$$ ✓

$$\sqrt[3]{(1+a^3)(1+b^3)(1+c^3)} \geq \left(\frac{ab+bc+ca+1}{2}\right)$$

$$(1+a^3)(1+b^3)(1+c^3) \geq \left(\frac{ab+bc+ca+1}{2}\right)^3$$

(C)



when $t=0$, $v=u$
when $t=T$, $v=0$
when $t=\frac{T}{2}$, $v=\frac{u}{4}$

$$ma = -m\lambda(c+v)$$

$$\frac{dv}{dt} = -\lambda(c+v) \qquad \text{1 mark}$$

$$\frac{dv}{c+v} = -\lambda\, dt$$

$$\ln(c+v) - \ln(c+u) = -\lambda t \qquad \text{1 mark}$$

$$\ln\left(\frac{c+v}{c+u}\right) = -\lambda t$$

when $t=\frac{T}{2}$, $v=\frac{u}{4}$

$$-\lambda \frac{T}{2} = \ln\left(\frac{c+u/4}{c+u}\right)$$

when $t=T$, $v=0$

$$-\lambda T = \ln\left(\frac{c+0}{c+u}\right)$$

$$\ln\left(\frac{c+u/4}{c+u}\right)^2 = \ln\left(\frac{c}{c+u}\right) \qquad \text{1 mark}$$

$$\left(\frac{c+u/4}{c+u}\right)^2 = \frac{c}{c+u}$$

$$\left(c+\frac{u}{4}\right)^2 = c(c+u)$$

$$c^2+\frac{cu}{2}+\frac{u^2}{16} = c^2+cu$$ ← 1 mark

$$\frac{u^2}{16} = \frac{cu}{2} \quad \therefore \quad c = \frac{u}{8}$$

(d) $t=0,\ x=m$

$m = a\cos\alpha$

$t=1,\ x=r$

$r = a\cos(n+\alpha)$

$t=2,\ x=r$

$r = a\cos(2n+\alpha)$

<u>Method 1:</u>

$$\begin{aligned} r+m &= a\cos(2n+\alpha)+a\cos\alpha \\ &= a(\cos(2n+\alpha)+\cos\alpha) \\ &= 2a\cos(n+\alpha)\cos n \\ &= 2r\cos n \end{aligned}$$

$$\cos n = \frac{r+m}{2r}$$

<u>Method 2:</u>

$$\begin{aligned} r &= a\cos(n+\alpha) \\ &= a\cos n\cos\alpha - a\sin n\sin\alpha \\ &= m\cos n - a\sin n\sin\alpha \end{aligned}$$

$\therefore\ a\sin n\sin\alpha = m\cos n - r$

$$\begin{aligned} r &= a\cos(2n+\alpha) \\ &= a(\cos 2n\cos\alpha - \sin 2n\sin\alpha) \\ &= m(2\cos^2 n-1) - 2a\sin n\cos n\sin\alpha \\ &= m(2\cos^2 n-1) - 2\cos n(m\cos n - r) \\ &= -m+2r\cos n \end{aligned}$$

$r+m = 2r\cos n$

$$\cos n = \frac{r+m}{2r}$$

Year 12 Extension 2 Trial HSC Question 15

a)

i.

$$ma = -(v^2 + v^3) \quad \text{(Newton's 2nd law)}$$

$$m = 1 \Rightarrow a = -(v^2 + v^3)$$

$$v\frac{dv}{dx} = -(v^2 + v^3)$$

$$\frac{dv}{dx} = -(v + v^2)$$

$$= -v(1 + v)$$

$$\int_{v_0}^{v} \frac{dv}{v(1+v)} = -\int_0^x dx$$

$$\int_{v_0}^{v} \left(\frac{1}{v} - \frac{1}{v+1}\right) dv = -x$$

$$-x = [\ln|v| - \ln|v+1|]_{v_0}^{v}$$

$$= \left[\ln\left|\frac{v}{v+1}\right|\right]_{v_0}^{v}$$

$$= \left[\ln\left(\frac{v}{v+1}\right)\right]_{v_0}^{v} \qquad (\text{Since } v > 0)$$

$$= \ln\left(\frac{v}{v+1}\right) - \ln\left(\frac{v_0}{v_0+1}\right)$$

$$= \ln\left(\frac{v(v_0+1)}{v_0(v+1)}\right)$$

$$x = \ln\left(\frac{v_0(v+1)}{v(v_0+1)}\right)$$

$$v = \frac{v_0}{2}, x = s$$

$$s = \ln\left(\frac{v_0\left(\frac{v_0}{2}+1\right)}{\frac{v_0}{2}(v_0+1)}\right)$$

$$= \ln\left(\frac{v_0+2}{v_0+1}\right) \text{ or } \ln\left(1 + \frac{1}{v_0+1}\right)$$

$$\text{Let } \frac{1}{v(v+1)} \equiv \frac{a}{v} + \frac{b}{v+1}$$

$$a(v+1) + bv \equiv 1$$

$$v = -1 \Rightarrow b = -1$$

$$v = 0 \Rightarrow a = 1$$

$$\therefore \frac{1}{v(v+1)} \equiv \frac{1}{v} - \frac{1}{v+1}$$

1st mark for getting to $v\frac{dv}{dx} = -(v^2 + v^3)$

2nd mark for correctly integrating the function with respect to v

3rd mark for getting the correct answer

ii.

$$x = \ln\left(\frac{v_0(1+v)}{v(1+v_0)}\right)$$

$$\frac{v_0(1+v)}{v(1+v_0)} = e^x$$

$$\frac{1+v}{v} = e^x\left(\frac{1+v_0}{v_0}\right)$$

$$1+\frac{1}{v} = e^x\left(\frac{1+v_0}{v_0}\right)$$

$$\frac{1}{v} = e^x\left(\frac{1+v_0}{v_0}\right) - 1 \qquad \left(v = \frac{v_0}{e^x(1+v_0)-v_0} = \frac{v_0e^{-x}}{(1+v_0)-v_0e^{-x}} = \frac{v_0e^{-x}}{1+v_0(1-e^{-x})}\right)$$

$$\frac{dt}{dx} = e^x\left(\frac{1+v_0}{v_0}\right) - 1$$

1st mark for correctly making $\frac{1}{v}$ or v the subject

$$\int_0^T dt = \int_0^s e^x\left(\frac{1+v_0}{v_0}\right) - 1\,dx$$

$$T = \left[e^x\left(\frac{1+v_0}{v_0}\right) - x\right]_0^s$$

2nd mark for correctly integrating with respect to x

$$= \left[e^s\left(\frac{1+v_0}{v_0}\right) - s\right] - \left(\frac{1+v_0}{v_0}\right)$$

$$= \left(\frac{v_0+2}{v_0+1}\right)\left(\frac{1+v_0}{v_0}\right) - s - \left(\frac{1+v_0}{v_0}\right)$$

$$= \left(\frac{v_0+2}{v_0}\right) - \left(\frac{1+v_0}{v_0}\right) - s$$

$$= \frac{1}{v_0} - s$$

3rd mark for correct answer

Alternate solution:

$$\frac{dv}{dt} = -(v^2+v^3)$$

$$= -v^2(1+v)$$

1st mark for $\frac{dv}{dt} = -(v^2+v^3)$

$$\int_{v_0}^{v} \frac{dv}{v^2(1+v)} = -\int_0^t dt$$

$$-t = \int_{v_0}^{v}\left(\frac{1}{v+1} - \frac{1}{v} + \frac{1}{v^2}\right)dv$$

$$= \int_{v_0}^{v}\left(\frac{1}{v+1} - \frac{1}{v}\right)dv + \int_{v_0}^{v}\left(\frac{1}{v^2}\right)dv$$

$$= x + \left[-\frac{1}{v}\right]_{v_0}^{v} \quad \text{(from part i.)}$$

2nd mark

$$= x + \frac{1}{v_0} - \frac{1}{v}$$

$$t = -x - \frac{1}{v_0} + \frac{1}{v}$$

$$t = T, v = \frac{v_0}{2}, x = s$$

$$T = -s - \frac{1}{v_0} + \frac{2}{v_0}$$

$$= \frac{1}{v_0} - s$$

3rd mark for correct answer

$$\frac{1}{v^2(v+1)} \equiv \frac{av+b}{v^2} + \frac{c}{v+1}$$

$$(av+b)(v+1) + cv^2 \equiv 1$$

$$v = -1 \Rightarrow c = 1$$

$$v = 0 \Rightarrow b = 1$$

Comparing coefficients of $v^2 \Rightarrow (a+c) = 0$

$$\therefore a = -1$$

$$\therefore \frac{1}{v^2(v+1)} \equiv \frac{1}{v+1} - \frac{1}{v} + \frac{1}{v^2}$$

iii.

$$t = -x - \frac{1}{v_0} + \frac{1}{v} \quad \text{(From iii)}$$ 1st mark for reference to t and x in terms of v and v_0

$$\frac{1}{v} = t + x + \frac{1}{v_0}$$

$$= \frac{v_0 t + v_0 x + 1}{v_0}$$

$$v = \frac{v_0}{v_0 t + v_0 x + 1}$$ 2nd mark for final answer

b)

i.

Terminal velocity happens when $a = 0$

$a = 0 \Rightarrow g - kv^2 = 0$

$kv^2 = g$

$v^2 = \frac{g}{k}$

$v = \sqrt{\frac{g}{k}}\ (v > 0)$ 1 mark

ii.

$$a = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$\frac{v}{g - kv^2}dv = dx$$ 1 mark for separating v and x for integration

$$-\frac{1}{2k}\int \frac{-2kv}{g - kv^2}dv = \int dx$$

$$x + c = -\frac{1}{2k}\ln|g - kv^2|$$ 1 mark correctly integrating both sides

$$= -\frac{1}{2k}\ln(g - kv^2) \quad (g - kv^2 > 0 \text{ as } a > 0\,)$$

$$x = 0, v = v_0 \Rightarrow c = -\frac{1}{2k}\ln(g - kv_0^2)$$

$$x - \frac{1}{2k}\ln(g - kv_0^2) = -\frac{1}{2k}\ln(g - kv^2)$$

$$x = \frac{1}{2k}\ln\left(\frac{g - kv_0^2}{g - kv^2}\right)$$ 1 mark for integration constant AND reason for removal of absolute value

$$\ln\left(\frac{g - kv_0^2}{g - kv^2}\right) = 2kx$$

$$\frac{g - kv_0^2}{g - kv^2} = e^{2kx}$$

$$g - kv^2 = (g - kv_0^2)e^{-2kx}$$

$$kv^2 = g - (g - kv_0^2)e^{-2kx}$$

$$v^2 = \frac{g}{k} - \left(\frac{g}{k} - v_0^2\right)e^{-2kx}$$

$$= V^2 - (V^2 - v_0^2)e^{-2kx}$$

$$= V^2 + (v_0^2 - V^2)e^{-2kx}$$ 1 mark for final answer

Potential wrong answers from Q15a)

i. Variation 1

$$\frac{1}{v(v-1)} \equiv \frac{a}{v} + \frac{b}{v-1}$$

$$1 \equiv a(v-1) + bv$$

$$v = 1 \Rightarrow b = 1$$

$$v = 0 \Rightarrow a = -1$$

$$\frac{1}{v(v-1)} \equiv \frac{1}{v-1} - \frac{1}{v}$$

$$\int_{v_0}^{\frac{v_0}{2}} \frac{1}{v(v-1)} dv = \int_0^s dx$$

$$\int_{v_0}^{\frac{v_0}{2}} \frac{1}{v-1} - \frac{1}{v} dv = \int_0^s dx$$

$$[\ln|v-1| - \ln|v|]_{v_0}^{\frac{v_0}{2}} = s$$

$$s = \left[\ln\left|\frac{v-1}{v}\right|\right]_{v_0}^{\frac{v_0}{2}}$$

$$= \ln\left|\frac{v_0-2}{v_0}\right| - \ln\left|\frac{v_0-1}{v_0}\right|$$

$$= \ln\left|\frac{v_0-2}{v_0-1}\right|$$

ii.

$$\frac{1}{v^2(v-1)} \equiv \frac{av+b}{v^2} + \frac{c}{v-1}$$

$$(av+b)(v-1) + cv^2 \equiv 1$$

$$v = 1 \Rightarrow c = 1$$

$$v = 0 \Rightarrow b = -1$$

Comparing coefficients of $v^2 \Rightarrow (a+c) = 0$

$\therefore a = -1$

$$\therefore \frac{1}{v^2(v-1)} \equiv \frac{1}{v-1} - \frac{1}{v} - \frac{1}{v^2}$$

$$\frac{dv}{dt} = -v^2 + v^3$$

$$\int_0^T dt = \int_{v_0}^{\frac{v_0}{2}} \left(\frac{1}{v-1} - \frac{1}{v} - \frac{1}{v^2}\right) dv$$

$$T = \left[\ln\left|\frac{v-1}{v}\right| + \frac{1}{v}\right]_{v_0}^{\frac{v_0}{2}}$$

$$= \ln\left|\frac{v_0\left(\frac{v_0}{2}-1\right)}{\frac{v_0}{2}(v_0-1)}\right| + \frac{1}{\frac{v_0}{2}} - \frac{1}{v_0}$$

$$= \ln\left|\frac{v_0-2}{v_0-1}\right| + \frac{1}{v_0}$$

$$= s + \frac{1}{v_0}$$

15b)

i.

ii.

$$x = \frac{1}{2k}\ln\left(\frac{g-kv^2}{g-kv_0^2}\right)$$

$$\ln\left(\frac{g-kv^2}{g-kv_0^2}\right) = 2kx$$

$$\frac{g-kv^2}{g-kv_0^2} = e^{2kx}$$

$$g-kv^2 = (g-kv_0^2)e^{2kx}$$

$$kv^2 = g-(g-kv_0^2)e^{2kx}$$

$$v^2 = \frac{g}{k} - \left(\frac{g}{k} - v_0^2\right)e^{2kx}$$

$$= V^2 - (V^2 - v_0^2)e^{2kx}$$

$$= V^2 + (v_0^2 - V^2)e^{2kx}$$

$$V^2 - v^2 = (V^2 - v_0^2)e^{-2kx}$$

$$-v^2 = -V^2 + (V^2 - v_0^2)e^{-2kx} \quad ①$$

$$v^2 = V^2 + (v_0^2 - V^2)e^{-2kx}$$

c) i) $\vec{AC} = 4\vec{AB}$

$z_3 - z_1 = 4(z_2 - z_1)$ ①

$z_3 = z_1 + 4z_2 - 4z_1$

$z^3 = 4z_2 - 3z_1$ ①

ii) $z_4 - z_3 \perp z_2 - z_1$

$z_4 - z_3 = ip(z_2 - z_1)$ for positive real p

$z_4 = z_3 + ip(z_2 - z_1)$ ①

$kz_2 = 4z_2 - 3z_1 + ip(z_2 - z_1)$

$k = 4 - \frac{3z_1}{z_2} + ip\left(1 - \frac{z_1}{z_2}\right)$

Let $z_1 = ae^{i\alpha}$ and $z_2 = be^{i\beta}$ and $\theta = \alpha - \beta$

$k = 4 - 3\frac{a}{b}e^{i(\alpha-\beta)} + ip\left(1 - \frac{a}{b}e^{i(\alpha-\beta)}\right)$

$= 4 - 3\frac{a}{b}e^{i\theta} + ip\left(1 - \frac{a}{b}e^{i\theta}\right)$

$= 4 - \frac{3a}{b}(\cos\theta + i\sin\theta) + ip\left(1 - \frac{a}{b}(\cos\theta + i\sin\theta)\right)$

$= 4 - \frac{3a}{b}\cos\theta + p\frac{a}{b}\sin\theta + i\left(p - \frac{ap}{b}\cos\theta - \frac{3a}{b}\sin\theta\right)$

Since k is real $\therefore p - \frac{a}{b}p\cos\theta - \frac{3a}{b}\sin\theta = 0$ ①

$$p = \frac{3a/b\,\sin\theta}{1 - \frac{a}{b}\cos\theta} = \frac{3a\sin\theta}{b - a\cos\theta}$$

Now

$k = 4 - \frac{3a}{b}\cos\theta + \frac{3a\sin\theta}{b - a\cos\theta} \times \frac{a}{b}\sin\theta$

$k = 4 - \frac{3a}{b}\cos\theta + \frac{3a^2\sin^2\theta}{b(b - a\cos\theta)}$

$= \frac{4(b^2 - ab\cos\theta) - 3a\cos\theta(b - a\cos\theta) + 3a^2\sin^2\theta}{b(b - a\cos\theta)}$

$= \frac{4b^2 - 4ab\cos\theta - 3ab\cos\theta + 3a^2\cos^2\theta + 3a^2\sin^2\theta}{b^2 - ab\cos\theta}$

$= \frac{4b^2 - 7ab\cos\theta + 3a^2}{b^2 - ab\cos\theta}$ ① working

$= \frac{4|z_2|^2 - 7|z_1||z_2|\cos\theta + 3|z_1|^2}{|z_2|^2 - |z_1||z_2|\cos\theta}$